

Particle with non-Abelian charge: classical and quantum

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ABSTRACT: We study the action for a non-Abelian charged particle in a non-Abelian background field in the worldline formalism, described by real bosonic variables, leading to the well known equations given by Wong. The isospin parts in the action can be viewed as the Lagrange multiplier term corresponding to a non-holonomic constraint restricting the isospins to be parallel transported. The path integration is performed over the isospin variables and as a result, the worldlines turn out to be constrained by the classical solutions for the isospins.

We derive a wave equation from the path integral, constructed as the constrained Hamiltonian operator acting on the wave function. The operator ordering corresponding to the quantum Hamiltonian is found and verified by the inverse Weyl transformation.

KEYWORDS: .

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1. Introduction

A starting point in the usual way of quantization for a system is to adopt its classical system as a fundamental framework. Interestingly, different from the case of an electrically charged particle, the dynamics of a particle with non-Abelian charge was quantized directly from analogy and extension of its quantum theory since a non-Abelian charged particle had not been observed in a classical context. If a classical dynamics of a particle is considered as the most fundamental basis to construct a quantum theory, to find a classical picture of a particle with non-Abelian charge can be not only an interesting quest but also a crucial part for understanding its quantum nature.

A set of classical equations of motion for a non-Abelian charged particle was first derived by Wong from the quantum action in which a non-Abelian charged particle is described by a Dirac field [1]. These are called Wong's equations and include the parallel

transport equation describing an isospin in addition to a non-Abelian extension of the Lorentz force equation.

The purpose of this paper is to find a classical picture for a non-Abelian charged particle in a background field. We start from a classical action for a non-Abelian charged particle and construct the path integral. Many actions producing the Wong equations have been proposed [2–5]. Since we are interested in a purely classical picture, we write the action using only variables valued in the real numbers. In particular, all our c -numbers commute with one another, so upon quantization they will become bosonic variables. In [2], using the constrained Hamiltonian formalism, a wave equation governed by the constrained Hamiltonian operator was written down though the operator orderings were unspecified. We find that the wave equation gets an additional term which breaks gauge invariance. This will be verified by an inverse Weyl transformation. The derivation brings forth certain issues of operator ordering.

As is usually done, for instance in [2, 3], we first consider the equations given by Wong as a starting point to eventually obtain a quantum description for a non-Abelian charged particle. The first step is to obtain the classical action producing these equations. As mentioned above, we use only commuting real variables for all degrees of freedom. Next we quantize this action using the constrained Hamiltonian formalism due to Dirac [6, 7]. Once the Hamiltonian is obtained, we construct the path integral in the worldline formalism, and derive the constrained wave equation. This equation is compared with the one derived in canonical quantization earlier, discussing an operator ordering issue [8].

2. Classical action for a non-Abelian charged particle

A classical point particle is characterized by its position x^μ in spacetime, and if it has non-Abelian charge, also by a vector in some internal space, corresponding to some representation of the gauge group. For now, we will take this vector I^a to be in the Lie algebra of the gauge group, and refer to it as the ‘isospin’ of the particle. We can also take the internal vector to be in the fundamental vector space of the gauge group, we will discuss that later.

Then the following well-known pair of equations describes the dynamics of a classical non-Abelian charged particle, first derived from the Dirac equation by Wong [1],

$$\dot{I}^a + g f_{abc} A_\mu^b I^c \dot{x}^\mu = 0, \quad (2.1)$$

$$m \frac{d}{dt} \left[\frac{\dot{x}_\mu}{(-\dot{x}^\nu \dot{x}_\nu)^{1/2}} \right] + g I^a F_{\mu\nu}^a \dot{x}^\nu = 0. \quad (2.2)$$

The equation of motion for the isospin $I^a(t)$ is a parallel transport equation along the trajectory of a particle, $x^\mu(t)$, with the connection $\Gamma_{ac\mu} = g f_{abc} A_\mu^b$. The gauge group will be taken to be compact semi-simple, and the structure constants f_{abc} will be chosen as real and totally antisymmetric. The equation of motion for $x^\mu(t)$ is a non-Abelian generalization of the Lorentz force equation. The parameter t is a parameter along the worldline of the particle, and a dot denotes differentiation with respect to t . Let us consider the following

action,

$$S = \int dt \left(-m(-\dot{x}^\mu \dot{x}_\mu)^{1/2} + J^a(\dot{I}^a + g f_{abc} A_\mu^b \dot{x}^\mu I^c) \right) \quad (2.3)$$

The variables appearing in this action take only real numbers as values, and will thus correspond to real bosonic variables after quantization. The background metric is $\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$. We have not included a term for the dynamics of the gauge field, as we ignore the backreaction of the charge on the field. The Euler-Lagrange equation obtained by varying $J^a(t)$ is the equation of motion for $I^a(t)$,

$$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{J}^a} - \frac{\partial L}{\partial J^a} = -\dot{I}^a - g f_{abc} A_\mu^b \dot{x}^\mu I^c. \quad (2.4)$$

Similarly, varying $I^a(t)$ gives a parallel transport equation for $J^a(t)$,

$$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{I}^a} - \frac{\partial L}{\partial I^a} = \dot{J}^a + g f_{abc} A_\mu^b \dot{x}^\mu J^c. \quad (2.5)$$

The equation of motion for $\dot{x}^\mu(t)$ is a non-Abelian version of the Lorentz force equation as may be expected,

$$m \frac{d}{dt} \left[\frac{\dot{x}_\mu}{(-\dot{x}^\nu \dot{x}_\nu)^{1/2}} \right] + g K^a F_{\mu\nu}^a \dot{x}^\nu = 0, \quad (2.6)$$

where $K^a = f_{abc} J^b I^c \equiv (J \times I)^a$. It is easy to check that K^a also satisfies the same parallel transport equation as for J^a and I^a . Therefore, the action Eq. (2.3) leads to the same set of equations Eq. (2.2) given by Wong.

$$\dot{K}^a + g f_{abc} A_\mu^b \dot{x}^\mu K^c = 0, \quad (2.7)$$

$$m \frac{d}{dt} \left[\frac{\dot{x}_\mu}{(-\dot{x}^\nu \dot{x}_\nu)^{1/2}} \right] + g K^a F_{\mu\nu}^a \dot{x}^\nu = 0. \quad (2.8)$$

We note that the action given here is essentially the same as the one given by Balachandran et al. [2], but written in terms of real variables I and J rather than the complex variable $\theta^a = I^a + iJ^a$.

2.1 Gauge invariance

The action is invariant under the following set of infinitesimal transformations,

$$\begin{aligned} \delta[I]^a &= g f_{abc} \Lambda^b I^c, \\ \delta[A]_\mu^a &= g f_{abc} \Lambda^b A_\mu^c - \partial_\mu \Lambda^a, \\ \Rightarrow \delta[F_{\mu\nu}^a] &= g f_{abc} \Lambda^b F_{\mu\nu}^c, \end{aligned} \quad (2.9)$$

where Λ^a are infinitesimal. The same transformation rule as for I^a applies to J^a and K^a . Under these transformations the parallel transport equation Eq. (2.4) transforms covariantly while the generalized Lorentz force equation Eq. (2.6) is invariant. For the

parallel transport equation,

$$\begin{aligned}
\delta[\dot{I}^a + gf_{abc}(A_\mu^b \dot{x}^\mu I^c)] &= \frac{d}{dt} \delta[I]^a + gf_{abc} \delta[A_\mu^b] \dot{x}^\mu I^c + gf_{abc} A_\mu^b \dot{x}^\mu \delta[I]^c \\
&= gf_{akl} \dot{\Lambda}^k I^l + gf_{akl} \Lambda^k \dot{I}^l + gf_{abc} (gf_{bkl} \Lambda^k A_\mu^l \dot{x}^\mu - \dot{\Lambda}^b) I^c \\
&\quad + gf_{abc} A_\mu^b \dot{x}^\mu (gf_{ckl} \Lambda^k I^l) \\
&= gf_{akl} \Lambda^k (\dot{I}^l + gf_{lbc} A_\mu^b \dot{x}^\mu I^c),
\end{aligned} \tag{2.10}$$

where we have used the Jacobi identity. Similarly, Eq. (2.6) is invariant under the gauge transformation since $F_{\mu\nu}^a$ and K^a transform covariantly.

The action has a further invariance under reparametrizations of the worldline. If t is replaced by some smooth function $\tau(t)$ with $\dot{\tau} \neq 0$, we must make the replacements

$$dt \rightarrow d\tau = \dot{\tau} dt, \quad \frac{d}{dt} \rightarrow \frac{d}{d\tau} = \frac{1}{\dot{\tau}} \frac{d}{dt}. \tag{2.11}$$

It is easy to see that the action (2.3) remains invariant.

3. Constrained Hamiltonian formalism

The Hamiltonian plays a central role in the time evolution of physical quantities. However, the presence of gauge symmetries makes this a constrained system — the variables in the Lagrangian are not all independent. These symmetries include invariance under the transformations of Eq. (2.9), as well as invariance under reparametrizations of the worldline. We will follow the treatment due to Dirac [6, 7], which we describe briefly here.

The Lagrangian is a function of (q, \dot{q}) . When the Hamiltonian is obtained by the Legendre transformation, a variable \dot{q} is converted to the canonical momentum p defined by $p = \frac{\partial L}{\partial \dot{q}}$. This definition may lead to some constraints, which are relations among the canonical variables, of the form $\phi_m(q, p) = 0$. These are added to the original Hamiltonian $H_0 = p\dot{q} - L$ with Lagrange multipliers $u^m(q, p)$, giving a new Hamiltonian,

$$H(q, p) = H_0(q, p) + u^m(q, p) \phi_m(q, p). \tag{3.1}$$

Next we require that the constraints should not change with time (the consistency conditions). These conditions can lead to new constraints as well as relations among the Lagrange multipliers.

$$0 = \dot{\phi}_m = [\phi_m, H_0] + u^n [\phi_m, \phi_n], \tag{3.2}$$

where the square brackets signify Poisson brackets. That is, these equations may give a new constraint (a secondary constraint) or specify unknown coefficients u^n .

(a) If a new constraint is found, it is again applied to the consistency condition in order to see whether it gives a new further constraint or restricts the coefficients u^n . This algorithm ends if no more new constraint comes out.

(b) If relations are found among the Lagrange multipliers, u^n , general solutions for u^n to Eq. (3.2) are written in the form $u^n = U^n + V^n$, where U^n are particular solutions

and hence specified, and V^n are homogeneous solutions, which are in a form of linear combination of the solutions with arbitrary coefficients v^a , $V^n = v^a V_a^n$.

Any constraint which has vanishing Poisson brackets with all constraints is called a first class constraint and all other constraints are called second class constraints. The consistency conditions ensure that the Hamiltonian has vanishing Poisson brackets with all the constraints. Ultimately all the constraints including second class constraints need to be imposed on the system in order to keep only independent variables in the theory.

In order to do this, we first define Dirac brackets using all the second class constraints, denoted by χ_α ,

$$[F, G]^* \equiv [F, G] - [F, \chi_\alpha] C^{\alpha\beta} [\chi_\beta, G], \quad (3.3)$$

where C is the matrix given by the matrix elements $C_{\alpha\beta} = [\chi_\alpha, \chi_\beta]$ and the inverse elements $C^{\alpha\beta} \equiv [C^{-1}]_{\alpha\beta}$. The Dirac bracket, like the Poisson bracket, is antisymmetric and satisfies the Jacobi identity, but the Dirac bracket of any quantity with a second class constraint is zero,

$$[\chi_\alpha, F]^* = 0. \quad (3.4)$$

This implies that by replacing the Poisson brackets by Dirac brackets $[F, G]^*$, we can set all constraints, including the second class constraints χ_α , equal to zero at the end of a calculation.

3.1 Worldline formalism

For technical convenience we will use the worldline formalism to write the action [9, 10], and then to compute the path integral, for the particle with non-Abelian charge. The action in Eq. (2.3) can be rewritten in the worldline formalism by introducing a Lagrange multiplier h ,

$$S = \int dt (L_0 + L_I), \quad (3.5)$$

where

$$L_0 = \frac{1}{2} h^{-1} \dot{x}^2 - h \frac{m^2}{2}, \quad L_I = J^a [\dot{I}^a + g(A_\mu \dot{x}^\mu \times I)^a]. \quad (3.6)$$

The introduction of h makes the action in the worldline formalism invariant under the reparametrizations

$$t \rightarrow f(t), \quad h \rightarrow h/(df/dt). \quad (3.7)$$

The equation of motion for h is

$$\dot{x}^2 + h^2 m^2 = 0, \quad (3.8)$$

while the other equations of motion are now

$$\dot{I}^a + g f_{abc} A_\mu^b I^c \dot{x}^\mu = 0, \quad (3.9)$$

$$\frac{d}{dt} (h^{-1} \dot{x}_\mu) + g K^a F_{\mu\nu}^a \dot{x}^\nu = 0, \quad (3.10)$$

where $K^a = f_{abc} J^b I^c \equiv (J \times I)^a$ as before. Using Eq. (3.8) in Eq. (3.10), we recover Eq. (2.2). Alternatively, if Eq. (3.8) is plugged back into the action, the previous Lagrangian

Eq. (2.3) is recovered. The canonical momenta can be calculated from Eq. (3.6),

$$P_J^a = 0, \quad (3.11)$$

$$P_I^a = J^a, \quad (3.12)$$

$$P_h = 0, \quad (3.13)$$

$$P_\mu = h^{-1}\dot{x}_\mu - gA_\mu^a K^a, \quad (3.14)$$

from which we can read off the constraints immediately,

$$\begin{aligned} \phi_1^a &= P_J^a \approx 0, \\ \phi_2^a &= P_I^a - J^a \approx 0, \\ \phi_h &= P_h \approx 0, \end{aligned} \quad (3.15)$$

where the symbol \approx indicated weak equality, i.e., equality on the submanifold defined by constraints. Now we write down the canonical Hamiltonian H_0 ,

$$\begin{aligned} H_0 &= P_\mu \dot{x}^\mu + P_I^a \dot{I}^a + P_J^a \dot{J}^a + P_h \dot{h} - (L_0 + L_I) \\ &= \frac{h}{2}[(P_\mu + gA_\mu^a K^a)^2 + m^2] + P_J^a \dot{J}^a + (P_I^a - J^a) \dot{I}^a + P_h \dot{h}, \end{aligned} \quad (3.16)$$

where it has been assumed that the time derivatives of the coordinates, e.g. \dot{I} , are expressed in terms of the canonical variables, in principle. If the constraints Eq. (3.15) are added to H_0 , the new Hamiltonian H_1 remains in the same form as before,

$$\begin{aligned} H_1 &= H_0 + C_1^a P_J^a + C_2^a (P_I^a - J^a) + C_h P_h \\ &= \frac{h}{2}[(P_\mu + gA_\mu^a K^a)^2 + m^2] + P_J^a [C_1^a + \dot{J}^a(p, q)] + (P_I^a - J^a) [C_2^a + \dot{I}^a(p, q)] \\ &\quad + P_h [C_h + \dot{h}(p, q)]. \end{aligned} \quad (3.17)$$

The consistency conditions require that the constraints have vanishing Poisson brackets with H_1 , so writing $\Pi_\mu = (P_\mu + gA_\mu^a K^a)$, we find

$$\begin{aligned} 0 &= \dot{\phi}_1^a = [\phi_1^a, H_1] \\ &= [C_2^a + \dot{I}^a(p, q)] + 2g(f_{abc} A_\mu^b I^c)(h/2)\Pi^\mu, \end{aligned} \quad (3.18)$$

$$\begin{aligned} 0 &= \dot{\phi}_2^a = [\phi_2^a, H_1] \\ &= -[C_1^a + \dot{J}^a(p, q)] - 2g(f_{abc} A_\mu^b J^c)(h/2)\Pi^\mu, \end{aligned} \quad (3.19)$$

$$\begin{aligned} 0 &= \dot{\phi}_h = [P_h, H_1] \\ &= -\frac{1}{2}[(P_\mu + gA_\mu^a K^a)^2 + m^2]. \end{aligned} \quad (3.20)$$

The first two relations allow us to eliminate $C_2^a + \dot{I}^a(p, q)$ and $C_1^a + \dot{J}^a(p, q)$ in H_1 . The last equation is a secondary constraint, which we denote by ϕ_3 ,

$$\phi_3 = (P_\mu + gA_\mu^a K^a)^2 + m^2 \approx 0. \quad (3.21)$$

Now we can rewrite H_1 with the unknown coefficients eliminated,

$$H_1 = \frac{h}{2}(\Pi^\mu \Pi_\mu + m^2) + (C_h + \dot{h})P_h + \frac{h}{2}[-2g(f_{abc}A_\mu^b J^c)\Pi^\mu P_J^a - 2g(f_{abc}A_\mu^b I^c)\Pi^\mu(P_I^a - J^a)], \quad (3.22)$$

where we have defined $\Pi_\mu = P_\mu + gA_\mu^a K^a$. We can see that ϕ_3 identically commutes with H_1 without giving any further constraint or condition for the coefficients,

$$\begin{aligned} \dot{\phi}_3 &= [\phi_3, H_1] \\ &= \frac{h}{2} \left(-2gf_{abc}A_\mu^b J^c \Pi^\mu [\Pi^2 + m^2, P_J^a] - 2gf_{abc}A_\mu^b I^c \Pi^\mu [\Pi^2 + m^2, P_I^a - J^a] \right) \\ &= \frac{h}{2} \left(-4gf_{abc}A_\mu^b J^c \Pi^\mu \Pi^\nu [\Pi_\nu, P_J^a] - 4gf_{abc}A_\mu^b I^c \Pi^\mu \Pi^\nu [\Pi_\nu, P_I^a] \right) = 0. \end{aligned} \quad (3.23)$$

Also, by the following consistency condition, C_h must vanish,

$$\dot{h} = [h, H_1] = (C_h + \dot{h})[h, P_h] = C_h + \dot{h}. \quad (3.24)$$

Thus the final expression for the Hamiltonian is

$$H = h \left[\frac{1}{2}(\Pi^\mu \Pi_\mu + m^2) - gf_{abc}A_\mu^b J^c P_J^a - gf_{abc}A_\mu^b I^c (P_I^a - J^a) \right] + \dot{h}P_h, \quad (3.25)$$

with the constraints

$$\phi_1^a = P_J^a \approx 0, \quad (3.26)$$

$$\phi_2^a = P_I^a - J^a \approx 0, \quad (3.27)$$

$$\phi_3 = (P_\mu + gA_\mu^a K^a)^2 + m^2 \approx 0, \quad (3.28)$$

$$\phi_h = P_h \approx 0. \quad (3.29)$$

Note that ϕ_1^a, ϕ_2^a and ϕ_3 can be recombined into one first class and two second class constraints. The term within the square brackets in Eq. (3.25) is the first class linear combination of these constraints. We may call this term ϕ_0 . Then we can choose ϕ_1^a and ϕ_2^a as the independent second class constraints, and ϕ_0 and ϕ_h as the independent first class constraints. In other words, the constraint in Eq. (3.28) is replaced by $\phi_0 \approx 0$. We note that the Hamiltonian of Eq. (3.25) is also a first class constraint by construction. If the gauge freedom in terms of arbitrary coefficients is considered, H does not explicitly show the gauge freedom for the first class secondary constraint ϕ_0 . For this, one can use the extended Hamiltonian, where the number of unknown coefficients explicitly matches with that of the first class constraints,

$$H_E = H + \frac{C_0}{2}\phi_0. \quad (3.30)$$

Recognizing \dot{h} is arbitrary we see that the number of the arbitrary coefficients is equal to that of the first class constraints.

We can calculate the Dirac brackets using the matrix

$$C_{12}^{ab} = \delta_{ab} = -C_{21}^{ab}, \quad (3.31)$$

and set the second class constraints to zero due to the property $[\chi_\alpha, F]^* = 0$ where χ_α is the second class constraint and F is an arbitrary quantity. The constraint $P_h \approx 0$, along with the gauge fixing condition $h = \lambda$, which we now set, allows us to remove a pair of variables (h, P_h) [9]. Then we find

$$H_E = C(\Pi^\mu \Pi_\mu + m^2), \quad (3.32)$$

as also given in [2]. One can see whether we use H or H_E the path integral will be in the same form as long as all the corresponding gauge conditions to the first constraints are applied for it.

4. Path integral

A general form of the path integral for a constrained system is given in [7, 11].

$$\int \mathcal{D}z^A \prod_{t,\alpha} \delta(\chi_\alpha) (\det[\chi_\alpha, \chi_\beta])^{1/2} \prod_{t,a} \delta(\phi_a) \delta(G_a) \prod_t \det[G_a, \phi_b] e^{\frac{i}{\hbar} S[z^a(t)]}, \quad (4.1)$$

where ϕ_a and χ_α are a first and a second class constraints, respectively and G_a is a gauge fixing condition for ϕ_a . Here $\delta(\chi_\alpha)$ is responsible for the second class constraints and $(\det[\chi_\alpha, \chi_\beta])^{1/2}$ for their Jacobian factor. After suitable gauge conditions for the first class constraints are chosen, a gauge condition and its corresponding first class constraint form a pair of second class constraints. Thus the delta functions corresponding to the first class constraint and the gauge fixing condition, $\delta(\phi_a)$ and $\delta(G_a)$, respectively, are similarly incorporated together with the Jacobian factor $\det[G_a, \phi_b]$ in the path integral. The gauge fixing constraints need to be introduced in order to fix the arbitrary coefficients on the corresponding first class constraints in the Hamiltonian. Applying the formula Eq. (4.1) to our case, we find that the path integral becomes

$$\int_0^\infty d\lambda \int \mathcal{D}x^\mu \mathcal{D}P_\mu \mathcal{D}I^a \mathcal{D}P_I^a \mathcal{D}h \delta[h - \lambda] [\phi_0, \chi_0] \delta[\phi_0] e^{\frac{i}{\hbar} \int dt \mathcal{L}_{in}|_{P_J^a=0, J^a=P_I^a}}. \quad (4.2)$$

The Jacobian factor $(\det[\chi_\alpha, \chi_\beta])^{1/2} = 1$ and the delta functions for the second class constraints have been used by setting P_J^a to zero and replacing J^a by P_I^a . The integration over the gauge choice λ for the first class constraint $P_h \approx 0$ implies that the gauge freedom, i.e., a reparametrization invariance, is factored out [9]. \mathcal{L}_{in} is

$$\begin{aligned} \mathcal{L}_{in} &= P_\mu \dot{x}^\mu + P_I^a \dot{I}^a + P_J^a \dot{J}^a + P_h \dot{h} - H \\ &= \Pi_\mu \dot{x}^\mu - g A_\mu^a \dot{x}^\mu K^a + P_I^a \dot{I}^a - h \phi_0. \end{aligned} \quad (4.3)$$

We already noted that the Hamiltonian itself is a constraint and hence zero. ϕ_0 can also be placed in the same footing. One can express the delta function for ϕ_0 in the following form

$$\delta[\phi_0] = \frac{\hbar}{2\pi} \int \mathcal{D}\omega e^{\frac{i}{\hbar} \int dt \omega \phi_0}. \quad (4.4)$$

Hence, the delta function for ϕ_0 can be combined into the Hamiltonian by redefining $h - \omega \rightarrow h$,

$$\begin{aligned}
\mathcal{D}h\delta[\phi_0]e^{\frac{i}{\hbar}\int\mathcal{L}_{in}(h)dt} &\sim \int \mathcal{D}h\mathcal{D}\omega e^{\frac{i}{\hbar}\int\mathcal{L}_{in}(\omega-h)dt} \\
&= \int \mathcal{D}h\mathcal{D}\omega e^{\frac{i}{\hbar}\int\mathcal{L}_{in}(h)dt} \\
&\sim \int \mathcal{D}h e^{\frac{i}{\hbar}\int\mathcal{L}_{in}(h)dt}
\end{aligned} \tag{4.5}$$

Therefore, including the delta function for the constraint ϕ_0 is redundant.

After the redefinition of h we put the gauge condition $\delta[h-\lambda]$. Then we notice that \mathcal{L}_{in} becomes different from the original Lagrangian $L_0 + L_I$, Eq. (3.6), only by $-\frac{h}{2}(\Pi - \frac{1}{h}\dot{x})^2$.

$$\begin{aligned}
\mathcal{L}_{in} &= P_\mu\dot{x}^\mu + P_I^a\dot{I}^a + P_J^a\dot{J}^a + P_h\dot{h} - H \\
&= \Pi_\mu\dot{x}^\mu - gA_\mu^a\dot{x}^\mu K^a + P_I^a\dot{I}^a - \frac{h}{2}\Pi^2 - \frac{h}{2}m^2 \\
&= -\frac{h}{2}\left(\Pi - \frac{1}{h}\dot{x}\right)^2 + \frac{1}{2h}\dot{x}^2 - \frac{h}{2}m^2 + J^a[\dot{I}^a + g(A_\mu\dot{x}^\mu \times I)^a] \\
&\quad + (P_I^a - J^a)\dot{I}^a + P_J^a\dot{J}^a.
\end{aligned} \tag{4.6}$$

where $\dot{x}^\mu \equiv \frac{x_{n+1}-x_n}{\Delta t}$, etc, so \mathcal{L}_{in} has been expressed only by canonical variables (p, q) , i.e., $\mathcal{L}_{in} = \mathcal{L}_{in}(p, q)$. After the terms $(P_I^a - J^a)\dot{I}^a$ and $P_J^a\dot{J}^a$ are removed by the second class constraints with the Dirac bracket formalism, the Lagrangian becomes as follows.

$$\mathcal{L}_{in} \Rightarrow -\frac{h}{2}\left(\Pi - \frac{1}{h}\dot{x}\right)^2 + \frac{1}{2h}\dot{x}^2 - \frac{h}{2}m^2 + P_I^a[\dot{I}^a + g(A_\mu\dot{x}^\mu \times I)^a]. \tag{4.7}$$

According to the definition of the momentum P_μ , the first term would be zero in continuum limit. However, one cannot simply set it equal to zero since $\dot{x}^\mu = \frac{x_{n+1}-x_n}{\Delta t}$ is treated as independent variables in the path integral. This implies that the path integral directly expressed from the given Lagrangian may be different from one reconstructed from the Hamiltonian. Our interest in this paper is in the integration over the internal variables (P_I^a, I^a) . In the next subsection we see that a certain gauge choice can allow us to integrate over the internal degrees of freedom.

4.1 Gauge condition for ϕ_0

Here we choose a gauge fixing condition χ_0 for ϕ_0 . The gauge condition comes into the path integral as the delta function $\delta[\chi_0]$ with the Jacobian $[\phi_0, \chi_0]$. We are interested in the case that the gauge fixing does not spoil form of the isospin part so that later we are able to integrate over the internal variables separately. We choose the gauge fixing condition,

$$\chi_0 = x^0 - t = 0. \tag{4.8}$$

It is easy to check the Poisson bracket $[\phi_0, \chi_0]$ is non-zero.

$$\begin{aligned}
[\phi_0, \chi_0] &= \left[\frac{1}{2}(\Pi^2 + m^2) + C_1^a \phi_1^a + C_2^a \phi_2^a, x^0 - t \right] \\
&= \left[\frac{1}{2}(\Pi^2 + m^2), x^0 - t \right] \\
&= \frac{1}{2} \times 2\Pi^\mu [P_\mu + gA_\mu^a K^a, x^0] \\
&= -\Pi^0.
\end{aligned} \tag{4.9}$$

By a change of variables $\Pi_\mu - \frac{1}{\hbar} \dot{x}_\mu \rightarrow P_\mu$, the Jacobian factor $[\phi_0, \chi_0]$ becomes $(-P^0 - \frac{1}{\hbar} \dot{x}^0)$. The relevant path integral can then be written as

$$\begin{aligned}
&\int \mathcal{D}h \delta[h - \lambda] \mathcal{D}x^0 \delta[x^0 - t] \mathcal{D}x^i \mathcal{D}P^i e^{\frac{i}{\hbar} \int dt \left\{ -\frac{\hbar}{2} P_i^2 + \frac{1}{2\hbar} \dot{x}_i^2 - \frac{\hbar}{2} m^2 + P_I^a [I^a + g(A_\mu \dot{x}^\mu \times I)^a] \right\}} \\
&\times \mathcal{D}P^0 \left[-P^0 - \frac{1}{\hbar} \dot{x}^0 \right] e^{\frac{i}{\hbar} \int dt \left[-\frac{\hbar}{2} (P_0)^2 + \frac{1}{2\hbar} (\dot{x}^0)^2 \right]} \\
&\sim - \int \mathcal{D}h \delta[h - \lambda] \mathcal{D}x^i e^{\frac{i}{\hbar} \int dt \left\{ \frac{1}{2\hbar} \dot{x}_\mu^2 - \frac{\hbar}{2} m^2 + P_I^a [I^a + g(A_\mu \dot{x}^\mu \times I)^a] \right\}_{x^0=t}},
\end{aligned} \tag{4.10}$$

where the Gaussian integrals over P^μ have been taken.

4.2 Integration over internal variables

In the last subsection we have chosen the gauge fixing condition for ϕ_0 . Since this gauge fixing does not change structure of the isospin part, we are able to compute an integration over isospin variables. Taking the relevant part of the action in a time-sliced form,

$$\begin{aligned}
S_I &= \sum_{n=1}^{N+1} P_{I,n}^a [I_n^a - I_{n-1}^a + g f_{abc} A_{\mu,n}^b (x_n^\mu - x_{n-1}^\mu) I_n^c] \\
&= \sum_{n=1}^{N+1} P_{I,n}^a (M_n^{ac} I_n^c - I_{n-1}^c),
\end{aligned} \tag{4.11}$$

where $M_n^{ac} \equiv \delta^{ac} + g f_{abc} A_{\mu,n}^b (x_n^\mu - x_{n-1}^\mu)$. The integral over $P_{I,n}^a$ in the path integral produces a delta function,

$$\int dP_{I,n}^a e^{\frac{i}{\hbar} P_{I,n}^a (M_n^{ac} I_n^c - I_{n-1}^c)} = 2\pi\hbar \delta(M_n^{ac} I_n^c - I_{n-1}^c). \tag{4.12}$$

What finally remains in the path integral is a product of all the delta functions and after integration over I^a it reduces to a single delta function,

$$\begin{aligned}
&(2\pi\hbar)^{N+1} \int dI_N^a \cdots dI_1^a \delta(M_{N+1} I_{N+1} - I_N) \delta(M_N I_N - I_{N-1}) \cdots \delta(M_1 I_1 - I_0) \\
&= (2\pi\hbar)^{N+1} \int dI_{N-1}^a \cdots dI_1^a \delta(M_N M_{N+1} I_{N+1} - I_{N-1}) \cdots \delta(M_1 I_1 - I_0) \\
&= (2\pi\hbar)^{N+1} \delta(M_1 \cdots M_N M_{N+1} I_{N+1} - I_0).
\end{aligned} \tag{4.13}$$

The matrix product $M_1 \cdots M_N M_{N+1}$ is a finite parallel transport operator backward in time, but it can be placed in time order. Let us define $(A_n)_{ac} = gf_{abc}A_{\mu,n}^b(x_n^\mu - x_{n-1}^\mu)/\Delta t_n$, where $\Delta t_n = t_n - t_{n-1}$. Then using the antisymmetry of A_n , we can write

$$\begin{aligned}
& [M_1 \cdots M_N M_{N+1}]_{ab} I_{N+1}^b \\
&= [(1 + A_1 \Delta t_1) \cdots (1 + A_{N+1} \Delta t_{N+1}) \cdots (1 + A_{N+1} \Delta t_{N+1})]_{ab} I_{N+1}^b \\
&= I_{N+1}^b [(1 - A_{N+1} \Delta t_{N+1}) \cdots (1 - A_{N+1} \Delta t_{N+1}) \cdots (1 - A_1 \Delta t_1)]_{ba} \\
&= \left[I_{N+1} T e^{-\int_{t_0}^{t_{N+1}} A dt} \right]^a, \tag{4.14}
\end{aligned}$$

The path integral for the isospin part thus becomes

$$\begin{aligned}
& \int dI_N^a \cdots dI_1^a \delta(M_{N+1} I_{N+1} - I_N) \delta(M_N I_N - I_{N-1}) \cdots \delta(M_1 I_1 - I_0) \\
& \propto \delta(I_f T e^{-\int_{t_i}^{t_f} A dt} - I_i). \tag{4.15}
\end{aligned}$$

We know that the solution of the classical equation $\dot{I}^a + gf_{abc}A_\mu^b \dot{x}^\mu = 0$ is

$$I_f = T e^{-\int_{t_i}^{t_f} A dt} I_i. \tag{4.16}$$

The unitary matrix $U = T e^{-\int_{t_i}^{t_f} A dt}$, which consists of real elements, has an inverse element $[U^{-1}]_{ab} = [U^\dagger]_{ab} = U_{ba}$. Thus, it can be seen that the solution of the parallel transport equation, Eq. (4.16) is equivalent to Eq. (4.15), that is,

$$\begin{aligned}
& I_f = U I_i \\
& \Rightarrow [U^{-1} I_f]^a = [I_i]^a \\
& \Rightarrow U_{ba} [I_f]^b = [I_f T e^{-\int_{t_i}^{t_f} A dt}]^a = [I_i]^a. \tag{4.17}
\end{aligned}$$

We see that the path integral agrees with the classical result. As a result it contributes as a constraint, the solution to the classical parallel transport equation. The path integral can be written in the form before the integration over the initial momentum $P_f^a(t_i) \equiv j^a$

$$\int_{-\infty}^{\infty} dj \int_0^{\infty} d\lambda \int \mathcal{D}h \delta(h - \lambda) \int \mathcal{D}x e^{\frac{i}{\hbar} \int dt [\frac{1}{2\hbar} \dot{x}^2 - \frac{\hbar}{2} m^2]} e^{\frac{i}{\hbar} j (I_f T e^{-\int_{t_i}^{t_f} A dt} - I_i)}, \tag{4.18}$$

where $A_{ac} \equiv gf_{abc}A_\mu^b \dot{x}^\mu$. Note that here we did not apply the gauge fixing for ϕ_0 from the subsection 4.1 because we will derive the Klein-Gordon constrained wave equation which has a gauge symmetry. In the next section and in Appendix B, the validity of this path integral will be supported by deriving the wave equation obtained as a constrained wave equation in the canonical quantization. In addition, Appendix A provides the derivation of the generalized Lorentz force equation Eq. (2.6) by varying the path integral.

5. Fundamental representation

So far we have chosen the internal degrees of freedom of the classical particle to be in the adjoint representation. It is not difficult to write an action with the internal degrees in the fundamental representation. In the action of Eq. (2.3) we first write the structure constants f_{abc} as $-i(T^b)_{ac}$ where T^b are the generators. Then the ‘charge vectors’ I and J are taken in the vector space of the fundamental representation, so that we can replace $f_{abc}A^bJ^aI^c \rightarrow -iA^bT_{ij}^bJ^iI^j$. If the fundamental representation is real, for example if the gauge group is $\text{SO}(N)$, we can choose the matrices T_{ij}^a to be antisymmetric and purely imaginary. Then the Lagrangian becomes

$$L = \frac{1}{2h}\dot{x}^2 - \frac{h}{2}m^2 + J^i(\dot{I}^i - igT_{ij}^aA_\mu^aI^j\dot{x}^\mu). \quad (5.1)$$

If the fundamental vector space is complex, for example when the gauge group is $\text{SU}(N)$, the vectors I and J will be complex. Then the Lagrangian of Eq. (5.1) doubles the internal degrees of freedom. In order to avoid this doubling, we replace in this Lagrangian

$$\begin{aligned} J^a \dot{I}^a &\rightarrow iI^{i*} \dot{I}^i \\ J^a f_{abc}A_\mu^b I^c &\rightarrow iI^{i*}(-iT_{ij}^b)A_\mu^b I^j, \end{aligned} \quad (5.2)$$

where now the T^a are Hermitian, but not all purely imaginary. Then we can write the Lagrangian as

$$L = \frac{1}{2h}\dot{x}^2 - \frac{h}{2}m^2 + iI^{i*}(\dot{I}^i - igT_{ij}^aA_\mu^aI^j\dot{x}^\mu). \quad (5.3)$$

We can then write the parallel transport equation for the complex charge vector I ,

$$\dot{I}^i - igT_{ij}^aA_\mu^aI^j\dot{x}^\mu = 0, \quad (5.4)$$

and the generalization of the Lorentz force equation is now

$$\begin{aligned} \frac{d}{dt} \left(\frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \right) + \frac{d}{dt} (gA_\mu^a T_{ij}^a I^{*i} I^j) - gT_{ij}^a I^{*i} I^j (\partial_\mu A_\nu^a) \dot{x}^\nu \\ = \frac{d}{dt} (h^{-1} \dot{x}_\mu) + gK^a F_{\mu\nu}^a \dot{x}^\nu = 0, \end{aligned} \quad (5.5)$$

where now we have defined $K^a = -T_{ij}^a I^{*i} I^j = K^{*a}$. The Lorentz force equation thus has the same form as Eq. (3.10), i.e. the same form for adjoint and fundamental representations. It can be easily seen that K^a satisfies the same parallel transport equation as in Eq. (2.7),

$$\begin{aligned} \dot{K}^a &= -T_{ij}^a (\dot{I}^{*i} I^j + I^{*i} \dot{I}^j) \\ &= igI^{*i} I^j (T^b T^a - T^a T^b)_{ij} A_\mu^b \dot{x}^\mu \\ &= -gf_{abc} A_\mu^b K^c \dot{x}^\mu. \end{aligned} \quad (5.6)$$

The constraints are similar to those in the adjoint representation,

$$\phi_1^i = P_{I^*}^i \approx 0, \quad (5.7)$$

$$\phi_2^i = P_I^i - iI^{*i} \approx 0, \quad (5.8)$$

$$\phi_h = P_h \approx 0. \quad (5.9)$$

We also note that

$$P_\mu = h^{-1}\dot{x}_\mu - gA_\mu^a K^a. \quad (5.10)$$

After removing the variables I^{*i} and $P_{I^*}^i$, similarly to Eq. (4.7), the Lagrangian in the fundamental representation, \mathcal{L}_{in}^f , can be written down,

$$\begin{aligned} \mathcal{L}_{in}^f &= P_\mu \dot{x}^\mu + P_I^i \dot{I}^a + P_{I^*}^i \dot{I}^{*i} + P_h \dot{h} - H \\ &= -\frac{h}{2} \left(\Pi - \frac{1}{h} \dot{x} \right)^2 + \frac{1}{2h} \dot{x}^2 - \frac{h}{2} m^2 + P_I^i \left(\dot{I}^i - ig T_{ij}^b A_\mu^b I^j \dot{x}^\mu \right). \end{aligned} \quad (5.11)$$

And the path integral becomes

$$\int_0^\infty d\lambda \int \mathcal{D}x^\mu \mathcal{D}P_\mu \mathcal{D}I^i \mathcal{D}P_I^i \mathcal{D}h \delta[h - \lambda] [\phi_0, \chi_0] \delta[\phi_0] e^{\frac{i}{\hbar} \int dt \mathcal{L}_{in}^f}. \quad (5.12)$$

6. Derivation of the constrained wave equation

In this section we derive the wave equation satisfied by the path integral. This links the classical formulation with the quantum one, in turn justifying the choice of the classical action. While we expect to find some generalization of the Klein-Gordon equation, the operator representation from the classical Hamiltonian brings ambiguity in ordering of operators [13]. We will see that the quantum Hamiltonian operator as derived from our path integral includes a term which does not remain invariant under the gauge transformations of Eq. (2.9). We will verify this result by taking an inverse Weyl transformation of the classical Hamiltonian. The derivation is long, so for the sake of clarity we have gathered some of the intermediate calculations in Appendix B.

The wave function at the final point (x_f, I_f, t_f) is the weighted sum of the wave functions at all possible starting positions (x_i, I_i, t_i) , weighted by the kernel

$$K(x_f, I_f, t_f; x_i, I_i, t_i) \equiv \langle q_f, t_f | q_i, t_i \rangle, \quad (6.1)$$

where $K(x_f, I_f, t_f; x_i, I_i, t_i)$ is the path integral Eq. (4.18) and q stands for all variables, both internal and external. We are interested in the differential equation satisfied by the wave function. So we consider an infinitesimal evolution with $t_i = t$ and $t_f = t + \Delta t$. We also write $x_f = x$, $I_f = I$ and $x_i = x - \xi$, $I_i = I - \eta$. Then we can write the wave function as

$$\psi(x, t + \Delta t) = N \int d\eta \int d\xi \psi(x - \xi, I - \eta, t) K(x, I, t + \Delta t; x - \xi, I - \eta, t), \quad (6.2)$$

where N is a proportionality constant to be determined by matching wave functions in different times. Since Δt is infinitesimal, we can write

$$\begin{aligned} \langle q_f, t + \Delta t | q_i, t \rangle &= \langle q_f | e^{-\frac{i}{\hbar} \hat{H} \Delta t} | q_i \rangle \\ &\sim \int dp e^{i(q_f - q_i)p/\hbar} e^{-\frac{i}{\hbar} H_c(p, (q_f + q_i)/2) \Delta t} \\ &\sim e^{\frac{i}{\hbar} \mathcal{L} \Delta t}, \end{aligned} \quad (6.3)$$

where \hat{H} is a Hamiltonian operator, $H_c(p, q)$ is a Hamiltonian function and \mathcal{L} is a Lagrangian in a discrete time interval. Here p stands for the momenta canonically conjugate to all variables. We identify the Lagrangian \mathcal{L} from Eq. (4.18) and expand $Te^{-\int_{t_i}^{t_f} Adt'}$ to first order in $\epsilon \equiv \lambda \Delta t$, using $[A]_{ac} = gf_{abc}A_\mu^b \dot{x}^\mu$ to write

$$\begin{aligned}\mathcal{L}\Delta t &= \left(\frac{1}{2\lambda}\dot{x}^2 - \lambda\frac{m^2}{2}\right)\Delta t + j\left(I_f Te^{-\int_{t_i}^{t_f} Adt'} - I_i\right) \\ &= \frac{1}{2\epsilon}(x_f - x_i)^2 - \frac{m^2}{2}\epsilon + j^a \left[I_f^b(\delta_{ba} - gf_{bca}A_\mu^c(x_f - x_i)^\mu) - I_i^a\right] + \mathcal{O}(\epsilon^2) \\ &= \frac{1}{2\epsilon}\xi^2 - \frac{m^2}{2}\epsilon + j^a \left(\eta^a + gf_{acb}A_\mu^c \xi^\mu I_f^b\right),\end{aligned}\quad (6.4)$$

where $\eta = I_f - I_i$ and $\xi = (x_f - x_i)$. According to the Weyl correspondence [13] the classical Hamiltonian H_c has to be evaluated at the midpoint $\frac{q_{j+1} + q_j}{2}$,

$$\begin{aligned}\langle q_{j+1} | e^{-i(t_{j+1} - t_j)\hat{H}/\hbar} | q_j \rangle &\propto \langle q_{j+1} | \int e^{-i(t_{j+1} - t_j)H_c(p, q)/\hbar} \Delta(p, q) dp dq | q_j \rangle \\ &\propto \int dp e^{\frac{i}{\hbar} \left[p \cdot \left(\frac{q_{j+1} - q_j}{t_{j+1} - t_j} \right) - H_c\left(p, \frac{q_{j+1} + q_j}{2}\right) \right] (t_{j+1} - t_j)},\end{aligned}\quad (6.5)$$

with

$$\Delta(p, q) = \int du e^{iq \cdot u/\hbar} |p - u/2\rangle \langle p + u/2|. \quad (6.6)$$

Thus the isospin dependent part of the action takes the midpoint value, denoted by a ‘bar’,

$$S_I = P_{I,n}^a [I_n^a - I_{n-1}^a + gf_{abc} \overline{A_{\mu,n}^b} I_n^c (x_n^\mu - x_{n-1}^\mu)]. \quad (6.7)$$

Using the expression for the path integral from Eq. (4.18) in Eq. (6.2) and using Eq. (6.4), we can write the wave function at $t + \Delta t$ as

$$\psi(x, t + \Delta t) = N \int d\eta \int d\xi \psi(x - \xi, I - \eta, t) K(\xi, \eta, t), \quad (6.8)$$

with

$$K(\xi, \eta, t) = \int_{-\infty}^{\infty} dj \int_0^{\infty} d\lambda e^{\frac{i}{\hbar} \left[\frac{1}{2\epsilon} \xi^2 - \epsilon \frac{m^2}{2} + j\eta + (\overline{A_\mu \times I}) \xi^\mu \right]}, \quad (6.9)$$

where

$$\begin{aligned}\overline{A_\mu \times I} \xi^\mu &= \frac{1}{2} [A_\mu(x) \times I + A_\mu(x - \xi) \times (I - \eta)] \xi^\mu \\ &= \left[A_\mu(x) \times I - \frac{1}{2} \xi^\nu \partial_\nu A_\mu(x) \times I - \frac{1}{2} A_\mu(x) \times \eta \right] \xi^\mu + \mathcal{O}(\xi^2 \eta).\end{aligned}\quad (6.10)$$

Up to the second order in ξ and the first order in η , the exponent in Eq. (6.9) can be expressed as

$$\begin{aligned}&\frac{i}{\hbar} \left[\frac{\xi^2}{2\epsilon} + j^a \left(A_\mu \times I - \frac{1}{2} A_\mu \times \eta \right)^a \xi^\mu - \frac{1}{2} j^a (\partial_\mu A_\nu \times I)^a \xi^\mu \xi^\nu + j^a \eta^a - \epsilon \frac{m^2}{2} \right] \\ &= \frac{i}{\hbar} \left(\frac{1}{2\epsilon} \bar{\xi}^\mu G_{\mu\nu} \bar{\xi}^\nu - \frac{\epsilon}{2} a^\mu G_{\mu\nu} a^\nu \right) + \frac{i}{\hbar} \left(j^a \eta^a - \epsilon \frac{m^2}{2} \right),\end{aligned}\quad (6.11)$$

where we have defined

$$\begin{aligned}
G_{\mu\nu} &\equiv \eta_{\mu\nu} - \frac{1}{2}\epsilon j^a[(\partial_\mu A_\nu \times I)^a + (\partial_\nu A_\mu \times I)^a], \\
a^\mu &\equiv G^{\mu\nu} j^a[A_\nu \times I - \frac{1}{2}A_\nu \times \eta]^a, \\
\bar{\xi}^\mu &\equiv \xi^\mu + \epsilon a^\mu,
\end{aligned} \tag{6.12}$$

and $G^{\mu\nu}$ is the matrix inverse of $G_{\mu\nu}$, so that $G_{\mu\nu}G^{\nu\lambda} = \delta_\mu^\lambda$. On the right hand side of Eq. (6.2), we expand the wave function $\psi(x - \xi, I - \eta, t)$ up to the second order in ξ and η ,

$$\begin{aligned}
\psi(x - \xi, I - \eta, t) &= \psi(x, I, t) - \xi^\mu \partial_\mu \psi - \eta^a \partial_a \psi + \xi^\mu \eta^a \partial_\mu \partial_a \psi \\
&\quad + \frac{1}{2} \xi^\mu \xi^\nu \partial_\mu \partial_\nu \psi + \frac{1}{2} \eta^a \eta^b \partial_a \partial_b \psi.
\end{aligned} \tag{6.13}$$

Inclusion of higher orders ξ and η in a wave function expansion brings higher order contribution in ϵ , as we will see below. Now we are ready to integrate to get the wave function at $t + \Delta t$,

$$\begin{aligned}
\psi(x, t + \Delta t) &= N \int d\lambda dj d\eta d\bar{\xi} \left[\psi - (\bar{\xi}^\mu - \epsilon a^\mu) \partial_\mu \psi - \eta^a \partial_a \psi + (\bar{\xi}^\mu - \epsilon a^\mu) \eta^a \partial_a \partial_\mu \psi \right. \\
&\quad \left. + \frac{1}{2} \eta^a \eta^b \partial_a \partial_b \psi + \frac{1}{2} (\bar{\xi}^\mu - \epsilon a^\mu) (\bar{\xi}^\nu - \epsilon a^\nu) \partial_\mu \partial_\nu \psi \right] \\
&\quad \times e^{[\frac{i}{2\epsilon} \bar{\xi}^\mu G_{\mu\nu} \bar{\xi}^\nu - i \frac{\epsilon}{2} a^\mu G_{\mu\nu} a^\nu - i \epsilon \frac{m^2}{2} + i j^a \eta^a] / \hbar}.
\end{aligned} \tag{6.14}$$

Let us first perform the Gaussian integral for $\bar{\xi}$. Odd order terms in $\bar{\xi}$ vanish, so those are dropped and only the terms up to first order in ϵ are kept in the result, which reads

$$\psi(x, t + \Delta t) = N \int d\lambda dj d\eta d\bar{\xi} I_\xi e^{[-i \frac{\epsilon}{2} a^\mu G_{\mu\nu} a^\nu - i \epsilon \frac{m^2}{2} + i j^a \eta^a] / \hbar}. \tag{6.15}$$

I_ξ is defined as

$$\begin{aligned}
I_\xi &\equiv \int d\bar{\xi} \left[\psi + \frac{1}{2} \bar{\xi}^\mu \bar{\xi}^\nu \partial_\mu \partial_\nu \psi + \epsilon a^\mu \partial_\mu \psi - \eta^a \partial_a \psi + \frac{1}{2} \eta^a \eta^b \partial_a \partial_b \psi - \epsilon a^\mu \eta^a \partial_a \partial_\mu \psi \right] \\
&\quad \times e^{\frac{i}{\hbar} \frac{1}{2\epsilon} \bar{\xi}^\mu G_{\mu\nu} \bar{\xi}^\nu} \\
&= N^{-1}(j) \left[\psi - \eta^a \partial_a \psi + \frac{1}{2} \eta^a \eta^b \partial_a \partial_b \psi \right. \\
&\quad \left. + \epsilon \left(a^\mu \partial_\mu \psi - a^\mu \eta^a \partial_a \partial_\mu \psi + \frac{i\hbar}{2} G^{\mu\nu} \partial_\mu \partial_\nu \psi \right) \right].
\end{aligned} \tag{6.16}$$

In this equation we have written

$$\begin{aligned}
N^{-1}(j) &= \int d\bar{\xi} e^{\frac{i}{\hbar} \frac{1}{2\epsilon} \bar{\xi}^\mu G_{\mu\nu} \bar{\xi}^\nu} = (2i\hbar\epsilon\pi)^{D/2} (\det G_{\mu\nu})^{-1/2}, \\
\int d\bar{\xi} \bar{\xi}^\mu \bar{\xi}^\nu e^{\frac{i}{\hbar} \frac{1}{2\epsilon} \bar{\xi}^\mu G_{\mu\nu} \bar{\xi}^\nu} &= i\hbar\epsilon N^{-1}(j) G^{\mu\nu}.
\end{aligned} \tag{6.17}$$

As mentioned earlier it can be seen that higher orders in ξ bring higher orders in ϵ . A similar statement is true about η . Next for integration for η and j let us keep terms only in the first order in ϵ ,

$$e^{-\frac{i}{\hbar}\epsilon a^\mu G_{\mu\nu} a^\nu} = 1 - \frac{i\epsilon}{2\hbar} \left[(j \cdot (A_\mu \times I))^2 + \frac{1}{4} (j \cdot (A_\mu \times \eta))^2 - j \cdot (A_\mu \times I) j \cdot (A^\mu \times \eta) \right] + O(\epsilon^2), \quad (6.18)$$

where we have used the fact that $G_{\mu\nu} = \eta_{\mu\nu} + O(\epsilon)$. Let us write the result as

$$\psi(x, t + \Delta t) = N \int_0^\infty d\lambda I e^{-i\epsilon \frac{m^2}{2\hbar}}, \quad (6.19)$$

where

$$\begin{aligned} I &= \int dj d\eta I_\xi e^{-\frac{i}{\hbar}\epsilon a^\mu G_{\mu\nu} a^\nu + \frac{i}{\hbar} j \cdot \eta} \\ &= \int dj d\eta N^{-1}(j) \left[\psi - \eta^a \partial_a \psi + \frac{1}{2} \eta^a \eta^b \partial_a \partial_b \psi \right] e^{\frac{i}{\hbar} j \cdot \eta} \\ &\quad + \epsilon \int dj d\eta N^{-1}(j) [-a^\mu \partial_\mu \psi + a^\mu \eta^a \partial_a \partial_\mu \psi] e^{\frac{i}{\hbar} j \cdot \eta} \\ &\quad - \epsilon \frac{i}{2\hbar} \int dj d\eta N^{-1}(j) \left[(j \cdot (A_\mu \times I))^2 + \frac{1}{4} (j \cdot (A_\mu \times \eta))^2 \right. \\ &\quad \quad \quad \left. - j \cdot (A_\mu \times I) j \cdot (A^\mu \times \eta) \right] \\ &\quad \quad \quad \times \left[\psi - \eta^a \partial_a \psi + \frac{1}{2} \eta^a \eta^b \partial_a \partial_b \psi \right] e^{\frac{i}{\hbar} j \cdot \eta} \\ &\quad + \frac{i\hbar\epsilon}{2} \int dj d\eta N^{-1}(j) G^{\mu\nu} \partial_\mu \partial_\nu \psi e^{\frac{i}{\hbar} j \cdot \eta} \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (6.20)$$

The integrals I_1, \dots, I_4 are calculated in Appendix B. Using the results given there, we can write the final result

$$\begin{aligned} I &= N^{-1}(0) \psi + \lambda N^{-1}(0) \frac{i\hbar\Delta t}{2} [-(\partial_\mu A^\mu \times I)^a \partial_a \psi - 2(A^\mu \times I)^a \partial_a \partial_\mu \psi \\ &\quad + \frac{1}{4} \text{Tr}(A \cdot A) \psi + (A^\mu \times I)^b f_{acb} A_\mu^c \partial_a \psi + (A_\mu \times I)^a (A^\mu \times I)^b \partial_a \partial_b \psi + \partial^2 \psi], \end{aligned} \quad (6.21)$$

where we have brought back λ using $\epsilon = \lambda \Delta t$, and written $\text{Tr}(A \cdot A) = f_{acb} A_\mu^c f_{bda} A^{d\mu}$. If all the terms, including the mass term $e^{-i\epsilon \frac{m^2}{2\hbar}}$, are now gathered, we find the wave equation with respect to the parameter t . Let us first choose the proportionality constant N such that

$$\begin{aligned} 1 &= N \int_0^\infty d\lambda N^{-1}(0) = N (i2\hbar\Delta t\pi)^{D/2} \int_0^\infty d\lambda \lambda^{D/2} \\ &= \lim_{\Lambda \rightarrow \infty} N \frac{(i2\hbar\Delta t\pi)^{D/2}}{D/2 + 1} \Lambda^{D/2+1}, \end{aligned} \quad (6.22)$$

where Λ is to be sent to infinity. That is,

$$N = \frac{D/2 + 1}{(i2\hbar\Delta t\pi)^{D/2}} \frac{1}{\Lambda^{D/2+1}}. \quad (6.23)$$

Going back to Eq. (6.21), we can write the wave function $\psi(x, I, t + \Delta t)$ at $t + \Delta t$ as

$$\begin{aligned} & \psi(x, I, t + \Delta t) - \psi(x, I, t) \\ &= \frac{i\hbar\Delta t}{2} N \int_0^\infty d\lambda \lambda N^{-1}(0) \left[-(\partial_\mu A^\mu \times I)^a \partial_a \psi - 2(A^\mu \times I)^a \partial_a \partial_\mu \psi + \frac{1}{4} \text{Tr}(A \cdot A) \psi \right. \\ & \quad \left. + (A^\mu \times I)^b f_{acb} A_\mu^c \partial_a \psi + (A_\mu \times I)^a (A^\mu \times I)^b \partial_a \partial_b \psi + \partial^2 \psi - \frac{m^2}{\hbar^2} \psi \right] \\ &= \frac{i\hbar\Delta t}{2} N \int_0^\infty d\lambda \lambda N^{-1}(0) \left[-(\partial_\mu A^\mu \times I)^a \frac{i}{\hbar} \hat{P}_I^a \psi - 2(A^\mu \times I)^a \left(\frac{i}{\hbar} \right)^2 \hat{P}_I^a \hat{P}_\mu \psi \right. \\ & \quad \left. + \frac{1}{4} \text{Tr}(A \cdot A) \psi + (A^\mu \times I)^b f_{acb} A_\mu^c \frac{i}{\hbar} \hat{P}_I^a \psi \right. \\ & \quad \left. + \left(\frac{i}{\hbar} \right)^2 (A_\mu \times I)^a (A^\mu \times I)^b \hat{P}_I^a \hat{P}_I^b \psi + \left(\frac{i}{\hbar} \right)^2 \hat{P}^2 \psi - \frac{m^2}{\hbar^2} \psi \right], \quad (6.24) \end{aligned}$$

where we have used that $\partial^2 = -\partial_0^2 + \partial_i^2 = -\frac{1}{\hbar^2}(-\hat{P}_0^2) - \frac{1}{\hbar^2}\hat{P}_i^2 = -\frac{1}{\hbar^2}\hat{P}^2$ and $\hat{P}_I^a = \frac{\hbar}{i}\partial_{I^a}$. The wave equation with respect to the worldline parameter t is

$$\begin{aligned} i\hbar \frac{\partial \psi}{\partial t} &= -\frac{1}{2} N \int_0^\infty d\lambda \lambda N^{-1}(0) \left[-i\hbar(\partial_\mu A^\mu \times I)^a \hat{P}_I^a + 2(A^\mu \times I)^a \hat{P}_I^a \hat{P}_\mu \right. \\ & \quad \left. + \frac{\hbar^2}{4} \text{Tr}(A \cdot A) + i\hbar(A^\mu \times I)^b f_{acb} A_\mu^c \hat{P}_I^a \right. \\ & \quad \left. - (A_\mu \times I)^a (A^\mu \times I)^b \hat{P}_I^a \hat{P}_I^b - \hat{P}^2 - m^2 \right] \psi \\ &= -\frac{1}{2} N \int_0^\infty d\lambda \lambda N^{-1}(0) \left[-(\hat{P}^2 + m^2) - 2(A_\mu^a \hat{K}^a) \hat{P}^\mu \right. \\ & \quad \left. - \eta^{\mu\nu} (A_\mu^a A_\nu^b) \hat{K}^a \hat{K}^b - i\hbar(\partial_\mu A^\mu \times I)^a \hat{P}_I^a \right. \\ & \quad \left. + \frac{\hbar^2}{4} \text{Tr}(A \cdot A) + i\hbar(A^\mu \times I)^b f_{acb} A_\mu^c \hat{P}_I^a \right] \psi \\ &= \frac{1}{2} N \int_0^\infty d\lambda \lambda N^{-1}(0) [D\psi + \Delta\psi], \quad (6.25) \end{aligned}$$

where

$$\hat{K}^a = -f_{bac} \hat{P}_I^b I^c, \quad (6.26)$$

$$D\psi = [(\hat{P}_\mu + gA_\mu^a \hat{K}^a)^2 + m^2]_{\hat{Q}\hat{Q}\dots\hat{P}\hat{P}} \psi, \quad (6.27)$$

$$\Delta\psi = \left[i\hbar(\partial_\mu A^\mu \times I)^a \hat{P}_I^a - \frac{\hbar^2}{4} \text{Tr}(A \cdot A) - i\hbar(A^\mu \times I)^b f_{acb} A_\mu^c \hat{P}_I^a \right] \psi. \quad (6.28)$$

We will see below that the operator $D + \Delta$ is an inverse Weyl transform (the Wigner transform) of the classical Hamiltonian $H = \frac{\hbar}{2}[(P_\mu + gA_\mu^a K^a)^2 + m^2]$. The notation $\hat{Q}\hat{Q}\dots\hat{P}\hat{P}$

means that all the momentum operators are put on the right while the position operators are on the left. It can be seen that the right hand side in Eq. (6.25) is divergent, since

$$\begin{aligned}
N \int_0^\infty d\lambda \lambda N^{-1}(0) &= N(i2\hbar\Delta t\pi)^{D/2} \int_0^\infty d\lambda \lambda^{D/2+1} \\
&= \frac{D/2+1}{(i2\hbar\Delta t\pi)^{D/2}} \frac{1}{\Lambda^{D/2+1}} \frac{(i2\hbar\Delta t\pi)^{D/2}}{D/2+2} \Lambda^{D/2+2} \\
&= \frac{D/2+1}{D/2+2} \Lambda.
\end{aligned} \tag{6.29}$$

Eq. (6.25) can be written as

$$i\hbar \frac{\partial}{\partial t} \psi(x^\mu, I^a, t) = \frac{1}{2} \frac{D/2+1}{D/2+2} \Lambda (D\psi + \Delta\psi). \tag{6.30}$$

Dividing the both sides by the divergent factor and taking the limit $\Lambda \rightarrow \infty$, we find

$$D\psi + \Delta\psi = \lim_{\Lambda \rightarrow \infty} i\hbar \frac{D/2+2}{D/2+1} \frac{1}{\Lambda} \frac{\partial}{\partial t} \psi(x^\mu, I^a, t) = 0. \tag{6.31}$$

This is the wave equation for ψ . For a fundamental representation, the wave equation is obtained by replacing, e.g., $P_I^a f_{abc} A_\mu^b I^c$ by $-iP_I^i T_{ij}^b A_\mu^b I^j$. Note that in this substitution one has to maintain the order of indices in f_{abc} . Then in the fundamental representation Eqs. (6.26)-(6.28) are replaced by

$$\hat{K}^a = iT_{ij}^a \hat{P}_I^i I^j, \tag{6.32}$$

$$D\psi = [(\hat{P}_\mu + igA_\mu^a T_{ij}^a I^j \hat{P}_I^i)^2 + m^2]_{\hat{Q}\hat{Q}\dots\hat{P}\hat{P}} \psi, \tag{6.33}$$

$$\Delta\psi = \left[-\hbar g (\partial_\mu A^{b\mu} T_{ij}^b I^j) \hat{P}_I^i + \frac{\hbar^2}{4} (T_{ij}^a T_{ji}^b A_\mu^a A^{b\mu}) + i\hbar g^2 (A^{a\mu} T_{ij}^a I^j) T_{ki}^b A_\mu^b \hat{P}_I^k \right] \psi. \tag{6.34}$$

6.1 Operator ordering

In this subsection we verify the wave equation of Eq. (6.31) by using a general mathematical formula which relates the quantum Hamiltonian operator to the Hamiltonian function in the path integral. The map from the operator to a function in the phase space is called the Weyl transformation. The inverse map is called the Wigner transformation. The correspondence between the two spaces is one-to-one. The Wigner transformation from a function $a(p, q)$ to an operator $A(\hat{P}, \hat{Q})$ is given in a compact form as [13, 14].

$$A(\hat{P}, \hat{Q}) = [e^{\frac{\hbar}{2i} \frac{\partial}{\partial p} \cdot \frac{\partial}{\partial q}} a(p, q)]_{p \rightarrow \hat{P}, q \rightarrow \hat{Q}; \hat{Q}\hat{Q}\dots\hat{P}\hat{P}}, \tag{6.35}$$

where $\hat{Q}\hat{Q}\dots\hat{P}\hat{P}$ means that all the momentum operators are placed on the right side while the position operators on the left side.

The classical Hamiltonian H used in the path integral was $\frac{\hbar}{2}[(P_\mu + gA_\mu^a K^a)^2 + m^2]$. The Wigner transformation of that function on the phase space should give us the appropriate operator to be used in the Schrödinger equation. We note that the terms in H for which

there is an ordering ambiguity are $2P^\mu A_\mu^a K^a$ and $\eta^{\mu\nu} A_\mu^a K^a A_\nu^b K^b$. Plugging these functions into Eq. (6.35), we get

$$e^{\frac{\hbar}{2i} \frac{\partial}{\partial P_\mu} \frac{\partial}{\partial x^\mu}} (2P^\mu A_\mu^a K^a) \Rightarrow 2A\hat{K}\hat{P} + \frac{\hbar}{i} \partial_\mu A^\mu \hat{K} \quad (6.36)$$

and

$$\begin{aligned} & e^{\frac{\hbar}{2i} \frac{\partial}{\partial P_I^a} \frac{\partial}{\partial I^a}} (\eta^{\mu\nu} A_\mu^a K^a A_\nu^b K^b) \\ & \Rightarrow (A_\mu \times I)^a (A^\mu \times I)^b \hat{P}_I^a \hat{P}_I^b \\ & \quad + A^a \cdot A^b \frac{\hbar}{2i} \frac{\partial}{\partial P_I^g} \frac{\partial}{\partial I^g} (f_{acd} f_{bef} P_I^c I^d P_I^e I^f) \\ & \quad - \frac{1}{2} A^a \cdot A^b \frac{\hbar^2}{4} \frac{\partial}{\partial P_I^g} \frac{\partial}{\partial I^g} \frac{\partial}{\partial P_I^h} \frac{\partial}{\partial I^h} (f_{acd} f_{bef} P_I^c I^d P_I^e I^f) \\ & = (A_\mu \times I)^a (A^\mu \times I)^b \hat{P}_I^a \hat{P}_I^b \\ & \quad + \frac{\hbar}{2i} (-2A_\mu^c f_{bca} (A^\mu \times I)^b) \hat{P}_I^a - \frac{1}{2} \frac{\hbar^2}{4} 2(f_{acb} A^c) \cdot (f_{bda} A^d) \\ & = (A_\mu \times I)^a (A^\mu \times I)^b \hat{P}_I^a \hat{P}_I^b - \frac{\hbar}{i} A_\mu^c f_{bca} (A^\mu \times I)^b \hat{P}_I^a - \frac{\hbar^2}{4} \text{Tr}(A \cdot A). \end{aligned} \quad (6.38)$$

The additional pieces, $\frac{\hbar}{2i} \partial_\mu A^\mu \hat{K}$, $-\frac{\hbar}{i} A_\mu^c f_{bca} (A^\mu \times I)^b \hat{P}_I^a$ and $-\frac{\hbar^2}{4} \text{Tr}[A \cdot A]$, exactly match with the terms in Eq. (6.28), the expression for $\Delta\psi$. Let us rewrite the expressions (6.36) and (6.38) in a different form. The operator (6.36) can be written as

$$e^{\frac{\hbar}{2i} \frac{\partial}{\partial P_\mu} \frac{\partial}{\partial x^\mu}} 2PAK \Rightarrow (\hat{P} \cdot A^a + A^a \cdot \hat{P}) K^a. \quad (6.39)$$

Also, using the relations

$$\begin{aligned} [(A_\mu \times I)^b, \hat{P}_I^a] &= g f_{bcd} A_\mu^c [I^d, \hat{P}_I^a] \\ &= g f_{bcd} A_\mu^c \delta^{da} = i\hbar g f_{bca} A_\mu^c, \end{aligned} \quad (6.40)$$

and

$$\begin{aligned} (A_\mu \times I)^a (A^\mu \times I)^b \hat{P}_I^a \hat{P}_I^b &= (A_\mu \times I)^a [(A_\mu \times I)^b, \hat{P}_I^a] \hat{P}_I^b + (A_\mu \times I)^a \hat{P}_I^a (A_\mu \times I)^b \hat{P}_I^b \\ &= (A_\mu \times I)^a (i\hbar f_{bca} A_\mu^c) \hat{P}_I^b + (A_\mu \times I)^a \hat{P}_I^a (A_\mu \times I)^b \hat{P}_I^b \\ &= (A_\mu \times I)^a (i\hbar f_{bca} A_\mu^c) \hat{P}_I^b + (A_\mu^a \hat{K}^a A^{b\mu} \hat{K}^b) \\ &= (A_\mu \times I)^b (i\hbar f_{acb} A_\mu^c) \hat{P}_I^a + (A_\mu^a \hat{K}^a A^{b\mu} \hat{K}^b) \\ &= -(A_\mu \times I)^b (i\hbar f_{bca} A_\mu^c) \hat{P}_I^a + (A_\mu^a \hat{K}^a A^{b\mu} \hat{K}^b), \end{aligned} \quad (6.41)$$

we can rewrite Eq. (6.38) as

$$\begin{aligned} e^{\frac{\hbar}{2i} \frac{\partial}{\partial P_I^a} \frac{\partial}{\partial I^a}} (\eta^{\mu\nu} A_\mu^a K^a A_\nu^b K^b) &\Rightarrow -(A_\mu \times I)^b (i\hbar f_{bca} A_\mu^c) \hat{P}_I^a + (A_\mu^a \hat{K}^a) (A^{b\mu} \hat{K}^b) \\ &\quad + i\hbar A_\mu^c f_{bca} (A^\mu \times I)^b \hat{P}_I^a - \frac{\hbar^2}{4} \text{Tr}(A \cdot A) \\ &= (A_\mu^a \hat{K}^a A^{b\mu} \hat{K}^b) - \frac{\hbar^2}{4} \text{Tr}(A \cdot A). \end{aligned} \quad (6.42)$$

Thus the Hamiltonian operator can be written without assigning a particular ordering,

$$\hat{H} = \frac{\hbar}{2} \left[(\hat{P}_\mu + g A_\mu^a \hat{K}^a)^2 + m^2 - g^2 \frac{\hbar^2}{4} \text{Tr}(A \cdot A) \right]. \quad (6.43)$$

We clearly see that the Hamiltonian operator consists of the operator naively replaced by the classical Hamiltonian and the gauge non-invariant term $g^2 \frac{\hbar^2}{4} \text{Tr}(A \cdot A)$. As far as the operators orderings are relevant, there is no reason to expect that the Hamiltonian function in the path integral should correspond to operator in which the phase space variables have been naively substituted by the corresponding quantum operators. We note that the last term has been previously found in the literature [12].

The Hamiltonian commutes with the internal angular momentum K^2 , so one can factor out the eigenfunction for K^2 . Because of using real and bosonic variables, K^a behaves as an ordinary angular momentum and hence has integer eigenvalues.

7. Conclusion

In this paper we have considered the question: What is a classical non-Abelian point particle? More specifically, what is the classical dynamics of such a particle in a background non-Abelian gauge field? We started from a classical action describing the position of the particle as well as its charge, described by a dynamical ‘internal’ vector in some representation of the gauge group. We found that when this internal vector is in either the adjoint or the fundamental representation, the charge vector that enters the generalized Lorentz force equation is an adjoint vector constructed from the original charge vector and its conjugate momentum. So the charge vector that determines the space-time trajectory of the particle is in the adjoint representation in both cases.

The equations were originally derived from quantum theory, so we decided to quantize the classical action in the path integral formalism as a kind of cross check. Using the worldline formalism so as to include particles of zero mass, we found that the sum over paths includes only those paths along which the internal charge vector is parallel transported. We also derived a wave equation from this path integral and showed that the Hamiltonian operator in the wave equation exactly matches with the Hamiltonian operator transformed from the classical Hamiltonian function by the Weyl correspondence.

There are however some differences between the quantum theory of a non-Abelian charged field and that of a non-Abelian point particle as constructed from the classical action. That is not a failure of quantization, nor a contradiction with established knowledge. The first point of departure is the fact that the classical ‘isospin’, the charge vector in the action, is a continuous variable, whereas for quantum particles, the isospin \vec{I} is quantized, with fixed \vec{I}^2 . This can be resolved in the following way. The charge vector \vec{K} in our setup corresponds to the isospin vector operator upon quantization. By construction the vector \vec{K} is like an angular momentum operator in the internal vector space, so it will have discrete eigenvalues when the theory is quantized. The quantum Hamiltonian commutes with \vec{K}^2 as can be easily checked, so a particle in a given isospin eigenstate remains in that eigenstate. Then the quantum particle with a fixed isospin corresponds to the particle in an

eigenstate of \vec{K}^2 . However, since \vec{K} corresponds to $\vec{x} \times \vec{p}$ in the internal space, it can have only integer eigenvalues. This is analogous to there being no truly classical description of half-integer spin. The solution would be to use anti-commuting variables for the internal charge vector. That would allow, for example, half-integer isospins when the gauge group is $SU(2)$.

Another discrepancy is the appearance of the term $\frac{1}{4}g^2\hbar^2\text{Tr}(A \cdot A)$ in the quantum Hamiltonian operator, Eq. (6.43). This term breaks the gauge symmetry present in the classical Hamiltonian, although the Hamiltonian is still symmetric under *constant* internal rotations. This term appears to be a genuine effect of quantization: the Hamiltonian operator as derived from the path integral exactly matches with the Hamiltonian operator constructed from the classical Hamiltonian function by the Weyl correspondence. While a Hamiltonian constructed directly from a non-Abelian gauge symmetric quantum field theory is not expected to contain such a term, we note that the anomalous term is not unknown in the literature [12].

The source of this discrepancy is the following. The constraint which implements gauge transformations comes from the Hamiltonian for the gauge field, but we have treated the non-Abelian gauge field A_μ as a background field, ignoring its dynamics. We could of course try to include the Lagrangian for the gauge field. But it is known that for an electrically charged point particle in a background electromagnetic field the joint action leads to inconsistencies, stemming from the fact that the field due to the charged particle itself diverges at the position of the latter [15]. The problem is resolved by starting from the Lorentz-Dirac equation instead of the Lorentz force equation, and making further modifications so that quantization leads to the Dirac equation [16, 17].

The appearance of the gauge symmetry breaking term in the quantum Hamiltonian for the non-Abelian point particle is related to this classical difficulty of defining the action of a point particle in a dynamical gauge field. In the Hamiltonian picture, the constraint which implements gauge symmetry appears only if the gauge fields have their own dynamics. But if the non-Abelian point particle is coupled to a dynamical gauge field, the radiation reaction must be included as for the ordinary electric charge. In that case, Wong's generalization of the Lorentz force equation will have to be replaced by something analogous to the Lorentz-Dirac equation, and a corresponding action, as the starting point.

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Appendices

A. Derivation of the equation of motion

In this appendix we show that by varying the path integral in Eq. (4.18) with respect to x^μ , we can recover the classical equation of motion for x^μ from the vanishing first order

variation,

$$0 = \delta \int_{-\infty}^{\infty} dj \int_0^{\infty} d\lambda \mathcal{D}h \delta(h - \lambda) \int \mathcal{D}x^\mu e^{\frac{i}{\hbar} \int dt [\frac{1}{2\hbar} \dot{x}^2 - h \frac{m^2}{2}] + \frac{i}{\hbar} j (I_f T e^{-\int_{t_i}^{t_f} A dt} - I_i)}, \quad (\text{A.1})$$

where $A_{ac} \equiv g f_{abc} A_\mu^b \dot{x}^\mu$. Using the relation

$$\delta U(t_b, t_a) = - \int_{t_a}^{t_b} dt' U(t_b, t') \delta A(t') U(t', t_a), \quad (\text{A.2})$$

where $U(t_b, t_a) = T e^{-\int_{t_a}^{t_b} A dt}$, we can write

$$\begin{aligned} \frac{\delta}{\delta x^\mu(t)} U(t_f, t_i) &= - \int_{t_i}^{t_f} dt' U(t_f, t') \frac{\delta}{\delta x^\mu(t)} A(t') U(t', t_i) \\ &= - \int_{t_i}^{t_f} dt' U(t_f, t') [\delta(t' - t) \partial_\mu A_\nu(t') \dot{x}^\nu(t') - A_\mu \frac{d}{dt} \delta(t' - t)] U(t', t_i) \\ &= -U(t_f, t) \partial_\mu A_\nu(t) \dot{x}^\nu U(t, t_a) - \frac{d}{dt} [-U(t_f, t) A_\mu U(t, t_i)] \\ &= -U(t_f, t) [\partial_\mu A_\nu(t) \dot{x}^\nu] U(t, t_i) \\ &\quad + U(t_f, t) A_\nu \dot{x}^\nu A_\mu U(t, t_i) + U(t_f, t) \partial_\nu A_\mu \dot{x}^\nu U(t, t_i) \\ &\quad - U(t_f, t) A_\mu A_\nu \dot{x}^\nu U(t, t_i) \\ &= -U(t_f, t) F_{\mu\nu} \dot{x}^\nu U(t, t_i), \end{aligned} \quad (\text{A.3})$$

where $[F_{\mu\nu}]_{ac} \equiv g f_{abc} [\partial_\mu A_\nu^b - \partial_\nu A_\mu^b + g f_{bde} A_\mu^d A_\nu^e]$. Recalling that it has been defined in Eq. (4.18) that $j^a = P_I^a(t_i)$, the relevant term $j I_f \frac{\delta}{\delta x^\mu(t)} U(t_f, t_i)$ can be written

$$\begin{aligned} P_I^a(t_i) [-I_f U(t_f, t) F_{\mu\nu} \dot{x}^\nu U(t, t_i)]_a &= P_I^a(t_i) [-I(t) F_{\mu\nu} \dot{x}^\nu U(t, t_i)]_a \\ &= [-I(t) F_{\mu\nu} \dot{x}^\nu U(t, t_i)]_a P_I^a(t_i) \\ &= -I(t) F_{\mu\nu} \dot{x}^\nu P_I(t) \\ &= -I^a(t) g f_{abc} F_{\mu\nu}^b \dot{x}^\nu P_I^c(t) \\ &= -g K^b F_{\mu\nu}^b \dot{x}^\nu, \end{aligned} \quad (\text{A.4})$$

where $K^a = f_{abc} P_I^b I^c$. Therefore, the vanishing first order in δx^μ in Eq. (A.1) leads to the equation of motion for x^μ .

$$\begin{aligned} &\int dt \left[-\frac{d}{dt} (h^{-1} \dot{x}_\mu) + J_i^a [-I_f U(t_f, t) F_{\mu\nu} \dot{x}^\nu U(t, t_i)]_a \right] \delta x^\mu \\ &= \int dt \left[-\frac{d}{dt} \left(\frac{m \dot{x}_\mu}{\sqrt{-\dot{x}^2}} \right) - g K^b F_{\mu\nu}^b \dot{x}^\nu \right] \delta x^\mu \\ &= \int dt \left[-\frac{d}{dt} \left(\frac{m \dot{x}_\mu}{\sqrt{-\dot{x}^2}} \right) - g K^b F_{\mu\nu}^b \dot{x}^\nu \right] \delta x^\mu \\ &\Rightarrow \frac{d}{dt} \left(\frac{m \dot{x}_\mu}{\sqrt{-\dot{x}^2}} \right) + g K^b F_{\mu\nu}^b \dot{x}^\nu = 0. \end{aligned} \quad (\text{A.5})$$

$$(\text{A.6})$$

B. Derivation of the constrained wave equation

The kernel was written from Eq. (6.11) in the following notation,

$$K(\xi, \eta, t) = \int_{-\infty}^{\infty} dj \int_0^{\infty} d\lambda \exp \left\{ \frac{i}{\hbar} \left[\frac{1}{2\epsilon} \bar{\xi}^\mu G_{\mu\nu} \bar{\xi}^\nu - \frac{\epsilon}{2} a^\mu G_{\mu\nu} a^\nu + j\eta - \epsilon \frac{m^2}{2} \right] \right\}, \quad (\text{B.1})$$

where

$$\begin{aligned} G_{\mu\nu} &\equiv \eta_{\mu\nu} - \frac{1}{2} \epsilon j^a [(\partial_\mu A_\nu \times I)^a + (\partial_\nu A_\mu \times I)^a], \\ a^\mu &\equiv G^{\mu\nu} j^a [A_\nu \times I - \frac{1}{2} A_\nu \times \eta]^a, \\ \bar{\xi}^\mu &\equiv \xi^\mu + \epsilon a^\mu, \end{aligned} \quad (\text{B.2})$$

The wave function $\psi(x, t + \Delta t)$ at $t + \Delta t$ is a sum of the wave functions $\psi(x - \xi, I - \eta, t)$ with the weight $K(\xi, \eta, t)$.

$$\begin{aligned} \psi(x, t + \Delta t) &= N \int d\eta \int d\xi \psi(x - \xi, I - \eta, t) K(\xi, \eta, t) \\ &= N \int d\lambda dj d\eta d\xi \left(\psi - \partial_\mu \psi \xi^\mu - \partial_a \psi \eta^a + \frac{1}{2} \partial_\mu \partial_\nu \psi \xi^\mu \xi^\nu \right. \\ &\quad \left. + \frac{1}{2} \partial_a \partial_b \psi \eta^a \eta^b + \partial_a \partial_\mu \psi \xi^\mu \eta^a \right) \\ &\quad \times e^{\frac{i}{2\hbar\epsilon} \bar{\xi}^\mu G_{\mu\nu} \bar{\xi}^\nu} e^{-\frac{i\epsilon}{2\hbar} a^\mu G_{\mu\nu} a^\nu - \frac{i\epsilon}{2\hbar} m^2 + \frac{i}{\hbar} j\eta} \\ &= N \int d\lambda dj d\eta d\bar{\xi} \left[\psi - \partial_\mu \psi (\bar{\xi}^\mu - \epsilon a^\mu) - \partial_a \psi \eta^a + \partial_a \partial_\mu \psi (\bar{\xi}^\mu - \epsilon a^\mu) \eta^a \right. \\ &\quad \left. + \frac{1}{2} \partial_a \partial_b \psi \eta^a \eta^b + \frac{1}{2} \partial_\mu \partial_\nu \psi (\bar{\xi}^\mu - \epsilon a^\mu) (\bar{\xi}^\nu - \epsilon a^\nu) \right] \\ &\quad \times e^{\frac{i}{2\hbar\epsilon} \bar{\xi}^\mu G_{\mu\nu} \bar{\xi}^\nu} e^{-\frac{i\epsilon}{2\hbar} a^\mu G_{\mu\nu} a^\nu - \frac{i\epsilon}{2\hbar} m^2 + \frac{i}{\hbar} j\eta} \\ &= N \int d\lambda dj d\eta d\bar{\xi} I_\xi e^{[-i\frac{\epsilon}{2} a^\mu G_{\mu\nu} a^\nu - i\epsilon \frac{m^2}{2} + i j^a \eta^a]/\hbar}, \end{aligned} \quad (\text{B.3})$$

which was Eq. (6.15). Let us perform the Gaussian integral I_ξ first. In the end we are only interested in the first orders in ϵ after suitable choice of normalization factor N .

$$\begin{aligned} I_\xi &= \int d\bar{\xi} \left[\psi - \partial_\mu \psi (\bar{\xi}^\mu - \epsilon a^\mu) - \partial_a \psi \eta^a + \frac{1}{2} \partial_\mu \partial_\nu \psi (\bar{\xi}^\mu - \epsilon a^\mu) (\bar{\xi}^\nu - \epsilon a^\nu) \right. \\ &\quad \left. + \frac{1}{2} \partial_a \partial_b \psi \eta^a \eta^b - \partial_a \partial_\mu \psi (\bar{\xi}^\mu - \epsilon a^\mu) \eta^a \right] e^{\frac{i}{2\hbar\epsilon} \bar{\xi}^\mu G_{\mu\nu} \bar{\xi}^\nu} \\ &= \int d\bar{\xi} \left[\psi - \partial_a \psi \eta^a + \frac{1}{2} \partial_a \partial_b \psi \eta^a \eta^b + \epsilon \partial_\mu \psi (a^\mu) - \epsilon \partial_a \partial_\mu \psi (a^\mu) \eta^a \right] e^{\frac{i}{2\hbar\epsilon} \bar{\xi}^\mu G_{\mu\nu} \bar{\xi}^\nu} \\ &\quad + \int d\bar{\xi} \left(\frac{1}{2} \partial_\mu \partial_\nu \psi \bar{\xi}^\mu \bar{\xi}^\nu \right) e^{\frac{i}{2\hbar\epsilon} \bar{\xi}^\mu G_{\mu\nu} \bar{\xi}^\nu} \\ &= N^{-1}(j) \left(\psi - \partial_a \psi \eta^a + \frac{1}{2} \partial_a \partial_b \psi \eta^a \eta^b \right) + N^{-1}(j) \epsilon [\partial_\mu \psi (a^\mu) - \partial_a \partial_\mu \psi (a^\mu) \eta^a] \\ &\quad + N^{-1}(j) \frac{i\hbar\epsilon}{2} G^{\mu\nu} \partial_\mu \partial_\nu \psi, \end{aligned} \quad (\text{B.4})$$

where it has been defined

$$N^{-1}(j) = \int d\bar{\xi} e^{\frac{i}{2\hbar\epsilon}\bar{\xi}^\mu G_{\mu\nu}\bar{\xi}^\nu} = (i2\hbar\epsilon\pi)^{D/2}(\det G_{\mu\nu})^{-1/2}. \quad (\text{B.5})$$

The vanishing exponential integrals with odd multiples in $\bar{\xi}$ have been dropped out and using the formulae $\delta(\det G_{\mu\nu}) = (\det G_{\mu\nu})G^{\mu\nu}\delta G_{\mu\nu}$ and $\delta G^{\mu\nu} = -G^{\mu\rho}G^{\nu\sigma}\delta G_{\rho\sigma}$, the following integral has been evaluated.

$$\begin{aligned} \int d\bar{\xi}\bar{\xi}^\mu\bar{\xi}^\nu e^{\frac{i}{2\hbar\epsilon}\bar{\xi}^\mu G_{\mu\nu}\bar{\xi}^\nu} &= -i2\hbar\epsilon \frac{\partial N^{-1}(j)}{\partial G_{\mu\nu}} \\ &= -i2\hbar\epsilon(i2\hbar\epsilon\pi)^{D/2} \frac{\partial G^{-1/2}}{\partial G_{\mu\nu}} \\ &= -i2\hbar\epsilon(i2\hbar\epsilon\pi)^{D/2} \left[-\frac{1}{2}(\det G_{\mu\nu})^{-3/2} \right] (\det G_{\mu\nu})G^{\mu\nu} \\ &= i\hbar\epsilon N^{-1}(j)G^{\mu\nu}. \end{aligned} \quad (\text{B.6})$$

Next what remains is to integrate over (η, j) .

$$I = \int dj d\eta I_\xi e^{-\frac{i\epsilon}{2\hbar}a^\mu G_{\mu\nu}a^\nu + \frac{i}{\hbar}j\eta}. \quad (\text{B.7})$$

Since only the first order in ϵ is needed, in the exponential $e^{-\frac{i\epsilon}{2\hbar}a^\mu G_{\mu\nu}a^\nu}$ only the terms up to first order in ϵ are kept.

$$\begin{aligned} &e^{-\frac{i\epsilon}{2\hbar}a^\mu G_{\mu\nu}a^\nu + \frac{i}{\hbar}j\eta} \\ &= \left[1 - \frac{i\epsilon}{2\hbar}j \left(A_\mu \times I - \frac{1}{2}A_\mu \times \eta \right) G^{\mu\nu}j \left(A_\nu \times I - \frac{1}{2}A_\nu \times \eta + \dots \right) \right] e^{\frac{i}{\hbar}j\eta} \\ &= \left[1 - \frac{i\epsilon}{2\hbar}j \left(A_\mu \times I - \frac{1}{2}A_\mu \times \eta \right) \eta^{\mu\nu}j \left(A_\nu \times I - \frac{1}{2}A_\nu \times \eta \right) + O(\epsilon^2) \right] e^{\frac{i}{\hbar}j\eta} \\ &= \left[1 - \frac{i\epsilon}{2\hbar} \left(j(A_\mu \times I)j(A^\mu \times I) + \frac{1}{4}j(A_\mu \times \eta)j(A^\mu \times \eta) - j(A_\mu \times I)j(A^\mu \times \eta) \right) \right] e^{\frac{i}{\hbar}j\eta} \\ &\quad + O(\epsilon^2), \end{aligned} \quad (\text{B.8})$$

where $G_{\mu\nu} = \eta_{\mu\nu} + O(\epsilon)$ has been used. The function $e^{\frac{i}{\hbar}j\eta}$ is proven to be useful since a function of η can be converted to a differential operator with respect to j .

$$\begin{aligned} \int dj d\eta e^{\frac{i}{\hbar}j\eta} f(\eta) g(j) &= \int dj d\eta g(j) f\left(\frac{\hbar}{i}\frac{\partial}{\partial j}\right) e^{\frac{i}{\hbar}j\eta} \\ &= \int dj d\eta e^{ij\eta} f\left(-\frac{\hbar}{i}\frac{\partial}{\partial j}\right) g(j) \\ &= (2\pi\hbar)^n \int dj \delta(j) f\left(-\frac{\hbar}{i}\frac{\partial}{\partial j}\right) g(j) \\ &= (2\pi\hbar)^n f\left(-\frac{\hbar}{i}\frac{\partial}{\partial j}\right) g(j)|_{j=0}, \end{aligned} \quad (\text{B.9})$$

where n in $(2\pi\hbar)^n$ is dimension of j^a . Note that we will omit $(2\pi\hbar)^n$ in later calculations since it can be absorbed into the normalization factor N . It is worthwhile to recognize two

properties for calculational convenience. The first is that to get non-vanishing terms the number of powers in η must be the same as that of j s. If $k \neq m$,

$$\int dj d\eta e^{\frac{i}{\hbar} j \eta} \eta^k j^m = 0 \quad (\text{B.10})$$

The second is that n th derivative of $N^{-1}(j)$ with respect to j brings ϵ^n order. For instance,

$$\begin{aligned} \frac{\partial N^{-1}(j)}{\partial j^a} &= (i2\epsilon\hbar\pi)^{D/2} \left[-\frac{1}{2}(\det G_{\mu\nu})^{-3/2} \right] (\det G_{\mu\nu}) G^{\mu\nu} \frac{\partial G_{\mu\nu}}{\partial j^a} \\ &= -\frac{1}{2} N^{-1}(j) G^{\mu\nu} \frac{\partial G_{\mu\nu}}{\partial j^a} = \frac{\epsilon}{2} N^{-1}(j) G^{\mu\nu} B_{\mu\nu}^a. \end{aligned} \quad (\text{B.11})$$

It can be easily seen that the further derivatives also have the similar properties. This implies that higher than first derivatives of $N^{-1}(j)$ with respect to j can be discarded because we want to keep terms only up to the first order in ϵ . The final integral I is written in four pieces.

$$I = \int dj d\eta I_\xi e^{-\frac{i\epsilon}{2\hbar} a^\mu G_{\mu\nu} a^\nu + \frac{i}{\hbar} j \eta} = I_1 + I_2 + I_3 + I_4, \quad (\text{B.12})$$

where

$$I_1 = \int dj d\eta N^{-1}(j) \left(\psi - \partial_a \psi \eta^a + \frac{1}{2} \partial_a \partial_b \psi \eta^a \eta^b \right) e^{\frac{i}{\hbar} j \eta}, \quad (\text{B.13})$$

$$I_2 = \epsilon \int dj d\eta N^{-1}(j) [\partial_\mu \psi(a^\mu) - \partial_a \partial_\mu \psi(a^\mu) \eta^a] e^{\frac{i}{\hbar} j \eta}, \quad (\text{B.14})$$

$$\begin{aligned} I_3 = -\frac{i\epsilon}{2\hbar} \int dj d\eta N^{-1}(j) &\left(j(A_\mu \times I) j(A^\mu \times I) + \frac{1}{4} j(A_\mu \times \eta) j(A^\mu \times \eta) \right. \\ &\quad \left. - j(A_\mu \times I) j(A^\mu \times \eta) \right) \\ &\times \left(\psi - \partial_a \psi \eta^a + \frac{1}{2} \partial_a \partial_b \psi \eta^a \eta^b \right) e^{\frac{i}{\hbar} j \eta}, \end{aligned} \quad (\text{B.15})$$

$$I_4 = \frac{i\hbar\epsilon}{2} \int dj d\eta N^{-1}(j) G^{\mu\nu} \partial_\mu \partial_\nu \psi e^{\frac{i}{\hbar} j \eta}. \quad (\text{B.16})$$

Let us evaluate the first integral I_1 .

$$\begin{aligned} I_1 &= \int dj d\eta N^{-1}(j) \left(\psi - \partial_a \psi \eta^a + \frac{1}{2} \partial_a \partial_b \psi \eta^a \eta^b \right) e^{\frac{i}{\hbar} j \eta} \\ &= \int dj d\eta N^{-1}(j) \left(\psi - \partial_a \psi \frac{\hbar}{i} \frac{\partial}{\partial j^a} \right) e^{\frac{i}{\hbar} j \eta}, \end{aligned} \quad (\text{B.17})$$

where $\frac{1}{2} \partial_a \partial_b \psi \eta^a \eta^b$ has been dropped out since it corresponds to the second derivative of $N^{-1}(j)$ with respect to j , which is in the second order in ϵ . Thus, using Eq. (B.11)

$$\begin{aligned} I_1 &= N^{-1}(0) \psi + \partial_a \psi \frac{\hbar}{i} \frac{\partial}{\partial j^a} N^{-1}(j)|_{j=0} \\ &= N^{-1}(0) \psi + \partial_a \psi \frac{\hbar\epsilon}{2i} N^{-1}(0) G^{\mu\nu} B_{\mu\nu}^a \\ &= N^{-1}(0) \left(\psi - \frac{i\hbar\epsilon}{2} \partial_a \psi \eta^{\mu\nu} B_{\mu\nu}^a \right) \\ &= N^{-1}(0) \left[\psi - \frac{i\hbar\epsilon}{2} (\partial_\mu A^\mu \times I)^a \partial_a \psi \right]. \end{aligned} \quad (\text{B.18})$$

Next, in I_2 we discard $j\eta^{\mu\nu}(A_\nu \times I)$ and $\frac{1}{2}j\eta^{\mu\nu}(A_\nu \times \eta)\eta^a$ since they do not have the same powers in j and η , so those terms vanish or contribute to higher order in ϵ .

$$\begin{aligned}
I_2 &= \epsilon \int dj d\eta N^{-1}(j) [\partial_\mu \psi(a^\mu) - \partial_a \partial_\mu \psi(a^\mu) \eta^a] e^{\frac{i}{\hbar} j \eta} \\
&= \epsilon \int dj d\eta N^{-1}(j) \left[\partial_\mu \psi j G^{\mu\nu} \left(A_\nu \times I - \frac{1}{2} A_\nu \times \eta \right) \right. \\
&\quad \left. - \partial_a \partial_\mu \psi j G^{\mu\nu} \left(A_\nu \times I - \frac{1}{2} A_\nu \times \eta \right) \eta^a \right] e^{\frac{i}{\hbar} j \eta} \\
&= \epsilon \int dj d\eta N^{-1}(j) \eta^{\mu\nu} \left[\partial_\mu \psi \left(-j \frac{1}{2} A_\nu \times \eta \right) - \partial_a \partial_\mu \psi (j (A_\nu \times I)) \eta^a \right] e^{\frac{i}{\hbar} j \eta} \\
&\quad + O(\epsilon^2),
\end{aligned} \tag{B.19}$$

where $G^{\mu\nu}$ has been replaced by $\eta^{\mu\nu}$ to keep only the first order in ϵ . And the first term in Eq. (B.19) does not contribute, i.e.,

$$\begin{aligned}
&\epsilon \int dj d\eta N^{-1}(j) \eta^{\mu\nu} \partial_\mu \psi \left(-j \left(\frac{1}{2} A_\nu \times \eta \right) \right) e^{\frac{i}{\hbar} j \eta} \\
&= -\epsilon \int dj d\eta \left[N^{-1}(j) \eta^{\mu\nu} \partial_\mu \psi \frac{\hbar}{i} \frac{\partial}{\partial j^a} \left(\frac{1}{2} A_\nu \times j \right)^a + \left(\frac{1}{2} A_\nu \times j \right)^a \frac{\hbar}{i} \frac{\partial}{\partial j^a} (N^{-1}(j) \eta^{\mu\nu}) \right] e^{\frac{i}{\hbar} j \eta} \\
&= 0 + O(\epsilon^2).
\end{aligned} \tag{B.20}$$

Finally, the only non-vanishing part in I_2 is

$$\begin{aligned}
I_2 &= -\epsilon \int dj d\eta N^{-1}(j) \partial_a \partial_\mu \psi \eta^{\mu\nu} (j (A_\nu \times I)) \eta^a e^{\frac{i}{\hbar} j \eta} \\
&= \epsilon \int dj d\eta N^{-1}(j) \partial_a \partial_\mu \psi \eta^{\mu\nu} \frac{\hbar}{i} \frac{\partial}{\partial j^a} (j (A_\nu \times I)) e^{\frac{i}{\hbar} j \eta} \\
&= \epsilon \int dj d\eta N^{-1}(j) \partial_a \partial_\mu \psi \eta^{\mu\nu} \frac{\hbar}{i} (A_\nu \times I)^a e^{\frac{i}{\hbar} j \eta} \\
&= \epsilon N^{-1}(0) \partial_a \partial_\mu \psi \frac{\hbar}{i} (A^\mu \times I)^a.
\end{aligned} \tag{B.21}$$

By the same reason as for I_2 , in I_3 only for the non-zero first order term in ϵ , only terms in the same power of η and j apart from $N^{-1}(j)$ are kept,

$$\begin{aligned}
I_3 &= -i \frac{\epsilon}{2\hbar} \int dj d\eta N^{-1}(j) \left(j (A_\mu \times I) j (A^\mu \times I) + \frac{1}{4} j (A_\mu \times \eta) j (A^\mu \times \eta) - j (A_\mu \times I) j (A^\mu \times \eta) \right) \\
&\quad \times \left(\psi - \partial_a \psi \eta^a + \frac{1}{2} \partial_a \partial_b \psi \eta^a \eta^b \right) e^{\frac{i}{\hbar} j \eta} \\
&= -i \frac{\epsilon}{2\hbar} \int dj d\eta N^{-1}(j) \left[\frac{1}{4} j (A_\mu \times \eta) j (A^\mu \times \eta) \psi - j (A_\mu \times I) j (A^\mu \times \eta) (-\partial_a \psi \eta^a) \right. \\
&\quad \left. + j (A_\mu \times I) j (A^\mu \times I) \left(\frac{1}{2} \partial_a \partial_b \psi \eta^a \eta^b \right) \right] e^{\frac{i}{\hbar} j \eta} \\
&= I_{31} + I_{32} + I_{33}.
\end{aligned} \tag{B.22}$$

We calculate I_3 in three parts, I_{31} , I_{32} and I_{33} .

$$\begin{aligned}
I_{31} &= -\frac{i\epsilon}{2\hbar} \int dj d\eta N^{-1}(j) \frac{1}{4} \psi j (A_\mu \times \eta) j (A^\mu \times \eta) e^{\frac{i}{\hbar} j \eta} \\
&= -\frac{i\epsilon}{2\hbar} \int dj d\eta N^{-1}(j) \frac{1}{4} \psi e^{\frac{i}{\hbar} j \eta} \frac{\hbar}{i} \frac{\partial}{\partial j^a} \frac{\hbar}{i} \frac{\partial}{\partial j^b} \left[(A_\mu \times j)^a (A^\mu \times j)^b \right] \\
&= \frac{i\epsilon\hbar}{2} N^{-1}(0) \frac{1}{4} (f_{acb} A_\mu^c) \eta^{\mu\nu} (f_{bea} A_\nu^e) \psi \\
&= \frac{i\epsilon\hbar}{8} N^{-1}(0) \text{Tr}(A \cdot A) \psi.
\end{aligned} \tag{B.23}$$

$$\begin{aligned}
I_{32} &= -\frac{i\epsilon}{2\hbar} \int dj d\eta N^{-1}(j) [(-j (A_\mu \times I))(j (A^\mu \times \eta))(-\partial_a \psi \eta^a)] e^{\frac{i}{\hbar} j \eta} \\
&= -\frac{i\epsilon}{2\hbar} \int dj d\eta N^{-1}(j) (-\partial_a \psi) e^{\frac{i}{\hbar} j \eta} \frac{\hbar}{i} \frac{\partial}{\partial j^a} \frac{\hbar}{i} \frac{\partial}{\partial j^b} [(-j (A_\mu \times I))(-A^\mu \times j)^b] \\
&= \frac{i\epsilon\hbar}{2} N^{-1}(0) (-\partial_a \psi) [-(A_\mu \times I)^b \eta^{\mu\nu} (+f_{acb} A_\nu^c)] \\
&= \frac{i\epsilon\hbar}{2} N^{-1}(0) (A^\mu \times I)^b f_{acb} A_\mu^c \partial_a \psi.
\end{aligned} \tag{B.24}$$

$$\begin{aligned}
I_{33} &= -\frac{i\epsilon}{2\hbar} \int dj d\eta N^{-1}(j) j (A_\mu \times I) j (A^\mu \times I) \left(\frac{1}{2} \partial_a \partial_b \psi \eta^a \eta^b \right) e^{\frac{i}{\hbar} j \eta} e^{\frac{i}{\hbar} j \eta} \\
&= -\frac{i\epsilon}{2\hbar} \int dj d\eta N^{-1}(j) \left(\frac{1}{2} \partial_a \partial_b \psi \right) \frac{\hbar}{i} \frac{\partial}{\partial j^a} \frac{\hbar}{i} \frac{\partial}{\partial j^b} [(j (A_\mu \times I))(j (A^\mu \times I))] \\
&= -\frac{i\epsilon\hbar}{2} N^{-1}(0) \left(\frac{1}{2} \partial_a \partial_b \psi \right) \frac{2}{-1} (A_\mu \times I)^a (A^\mu \times I)^b \\
&= \frac{i\epsilon\hbar}{2} N^{-1}(0) (A_\mu \times I)^a (A^\mu \times I)^b \partial_a \partial_b \psi.
\end{aligned} \tag{B.25}$$

The last integral I_4 is straightforward.

$$I_4 = \frac{i\epsilon\hbar}{2} \int dj d\eta N^{-1}(j) G^{\mu\nu} \partial_\mu \partial_\nu \psi e^{\frac{i}{\hbar} j \eta} = \frac{i\epsilon\hbar}{2} N^{-1}(0) \partial^2 \psi. \tag{B.26}$$

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